ORIGINAL PAPER

Computer modeling of synthesis of calcium hydroxyapatite (CHAp)

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Abstract   In our previous papers (Mackevičius et al. in Cent Eur J Chem 10(2):380–385, 2012 , J Math Chem 50(8):2291–2302, 2012), we presented a method for estimation of the diffusion and reaction rates of synthesis at high temperatures using limited information, such as synthesis time and dimensions of reactants, from real laboratory experiments. The method was limited to the two-reactant case. In order to extend the method to the three-reactant case, the form and distribution of particles of three reactants must satisfy requirements of periodicity and symmetry. In our model, we achieve this by taking rhombic particles and a triangular synthesis space. Solving in the latter an inverse modeling problem, we obtain explicit formulas for the diffusion coefficient and reaction rate as functions of temperature by calculating the activation energies and other parameters of CHAp synthesis.

Keywords   Sol–gel processing · Three-reactant model · Diffusion coefficient · Reaction rate

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1 Introduction

The rate of a general homogeneous reaction is conventionally measured by following the decrease in concentration of reactants or the increase in concentration of the products at constant temperature. For the heterogenous reaction, however, the concept of concentration no longer has the same significance, and the progress of reaction is usually determined in some other way. The kinetics and the mechanism studies of heterogeneous reactions thus involve measurement of changes in mass of reactants of the sample as functions of time at constant temperatures [3–6]. Many equations relating the rate of solid-state reactions under isothermal and nonisothermal conditions to the bulk nucleation followed by three-dimensional growth and diffused distribution at the phase boundary (classical Fick diffusion or Prout–Tompkins model) are summarized and discussed in the literature [7–11]. The interpretation of the kinetic equations is extremely complicated and considers the way in which the reaction starts, by a process of nucleation, then how those nuclei grow and what reaction or interface geometry is involved, and finally, how the reactants decay .Consequently, for the investigation of complex solid-state reactions different novel mathematical approximations and computational models recently have been suggested [12–14].

Synthetic calcium hydroxyapatite, $Ca_{10}(PO_4)_6(OH)_2$ (CHAp for short) is known to be one of the most important implantable materials due to its biocompatibility, bioactivity, and osteoconductivity and is used as a substitute material for human hard tissues [15–17]. However, the specific chemical, structural, and morphological properties of CHAp bioceramics are highly sensitive to the processing conditions [18–21]. Several sol–gel approaches starting from nonaqueous [22,23] and aqueous [24,25] solutions of different precursors of calcium and phosphorus have been used for the preparation of CHAp powders. The aqueous route of sol–gel preparation of calcium hydroxyapatite offers an effective and relatively simple sol–gel procedure for CHAp. The aqueous sol–gel process offers considerable advantages of good mixing of the starting materials and excellent chemical homogeneity of the product. However, it has been well demonstrated that many parameters of sol–gel process such as starting materials and its concentration, pH, temperature, speed of stirring, stirring time, duration of gelation, nature and concentration of complexing agents, and others should be carefully controlled [26–29]. It was shown that these parameters in the aqueous sol–gel processing have a huge impact on the phase purity, morphological and other physical properties of the end products. The obtained results initiated the continuation of this work to investigate the possibility of computational modeling on the formation of sol–gel-derived calcium hydroxyapatite.

We applied our model for estimation of the parameters of the CHAp synthesis, such as the diffusion coefficients and reaction rates, using a rather limited information from real laboratory experiments. Namely, the known data only includes the synthesis half-times at different temperatures and approximate sizes of reactant particles. We use the Fick, Arrhenius, and active mass laws. Though the validity of applying the Arrhenius law to heterogeneous reactions has been questioned, the parameters obtained from it often have practical values even if their theoretical interpretation is difficult.

In this paper, we consider the two-dimensional (in space variables) model. In the two-reactant case [1,2], to simplify the calculations, we assumed that the square parti-

cles are distributed periodically in the chessboard order. This allowed us to reduce the calculations in the square-form synthesis space such that the whole synthesis space is symmetric with respect to the boundary (the edges of the square). To extend the model to the three-reactant case, the form and distribution of the particles of three reactants must also satisfy appropriate requirements of periodicity and symmetry. In our model, we achieve this by taking rhombic particles and reducing calculations to a triangular synthesis space. Solving in the latter an inverse modeling problem, we obtain explicit formulas for the diffusion coefficient and reaction rate as functions of temperature by calculating the activation energies and other parameters of CHAp synthesis.

The paper is organized as follows. In Sect. 2, we give a brief description of laboratory experiments of CHAp synthesis. In Sect. 3, we present a mathematical model for estimation of diffusion–reaction synthesis rates. In Sect. 4, we formulate the main goal of the paper, the parameters to be estimated. In Sect. 5, we present the calculation method and steps and, in Sect. 6, the results obtained. We conclude in Sect. 7.

2 Experimental

In the sol–gel process phosphoric acid, H_3PO_4, and calcium acetate monohydrate, $Ca(CH_3COO)2H_2O$, were selected as P and Ca precursors, respectively. Calcium acetate was first dissolved in 0.2 M CH_3COOH at room temperature (20 °C) or at 65 °C. To these solutions, phosphoric acid was added and the resulting mixtures were stirred for 1 h at the same temperatures. In a following step, tartaric acid as complexing agent was added to the above solutions. After concentrating the solutions by slow evaporation at 20 °C or 65 °C under stirring the Ca–P–O sols turned into the transparent gels. The oven-dried (100 °C) gel powders were ground in an agate mortar and annealed for 5 h at 800–1,300 °C in air.

3 Mathematical model

We denote by C_4 the concentration of the complex $Ca_{10}(PO_4)_6(OH)_2$ resulting from the synthesis of three complexes 10Ca, 6PO$_4$, and 2OH with concentrations C_1, C_2 and C_3, respectively, in the reaction

$$10Ca^{2+} + 6PO_4{}^{3-} + 2OH^- \rightarrow Ca_{10}(PO_4)_6(OH)_2. \tag{1}$$

The reaction rate w can be expressed by the rate law as follows:

$$w = kC_1C_2C_3, \tag{2}$$

where k is the reaction rate.

The dynamics of concentrations of the reactants by diffusion is described by the second Fick's law:

$$w_i = \frac{\partial C_i}{\partial t} = \sum_{j=1}^{2} D_i \frac{\partial^2 C_i}{\partial x_j^2}, \tag{3}$$

where D_i is the diffusion coefficient, n is the dimension of the model, and w_i is the rate of synthesis of the ith reactant. We denote

$$c_1 = 10C_1, \ c_2 = 6C_2, \ c_3 = 2C_3, \ c_4 = C_4; \tag{4}$$

that is, $c_i = c_i(x, t)$ is the concentration of the ith reactant (Ca, PO$_4$, OH, Ca$_{10}$(PO$_4$)$_6$(OH)$_2$ for $i = 1, 2, 3, 4$, respectively) of the synthesis at a point $x \in V$ at time t.

Using the second Fick law and the active mass law for the reaction (1), we get the following partial differential equation system on the synthesis space V:

$$\frac{\partial c_1}{\partial t} = \sum_{j=1}^{2} D_1 \frac{\partial^2 c_1}{\partial x_j^2} - \frac{1}{12} k c_1 c_2 c_3, \tag{5}$$

$$\frac{\partial c_2}{\partial t} = \sum_{j=1}^{2} D_2 \frac{\partial^2 c_2}{\partial x_j^2} - \frac{1}{20} k c_1 c_2 c_3, \tag{6}$$

$$\frac{\partial c_3}{\partial t} = \sum_{j=1}^{2} D_3 \frac{\partial^2 c_3}{\partial x_j^2} - \frac{1}{60} k c_1 c_2 c_3, \tag{7}$$

$$\frac{\partial c_4}{\partial t} = \sum_{j=1}^{2} D_4 \frac{\partial^2 c_4}{\partial x_j^2} + \frac{1}{120} k c_1 c_2 c_3, \tag{8}$$

with initial conditions $c_i(x, 0) = c_i^0(x)$, $x = (x_1, x_2) \in \overline{V} = V \cup \partial V$, $i = 1, 2, 3, 4$, and some boundary conditions (to be stated below) on the boundary ∂V of V. Since the sizes of particles of Ca, PO$_4$, OH, and Ca$_{10}$(PO$_4$)$_6$(OH)$_2$ are similar, we assume that all the diffusion coefficients coincide, that is, $D_1 = D_2 = D_3 = D_4 = D$.

We analyze the relations between D and k by using the following data known from the true laboratory experiments at Vilnius University: the synthesis times at different temperatures and typical dimensions of the reactant particles. In our model, for calculations, we use the synthesis half-time, which is the time, denoted $t_{1/2}$, in which the total concentration of initial reactants falls to one half; so, it satisfies the equation

$$\int_V (c_1(x, t_{1/2}) + c_2(x, t_{1/2}) + c_3(x, t_{1/2})) dx$$

$$= \frac{1}{2} \int_V (c_1(x, 0) + c_2(x, 0) + c_3(x, 0)) dx. \tag{9}$$

Theoretically, the total concentration of initial reactants, although decays exponentially, always remains positive, and therefore the "full" synthesis time is infinite. Practically, we assume that the synthesis is over when the total concentration of initial reactants becomes "sufficiently small," namely, the unreacted part is less than $0.1\% \approx 2^{-10}$. From experiments we know that the "full" synthesis times are approximately 6, 8, 10, and 16 h at the temperatures $T = 1{,}200, 100, 1{,}000,$ and $900\,°\mathrm{C}$, respec-
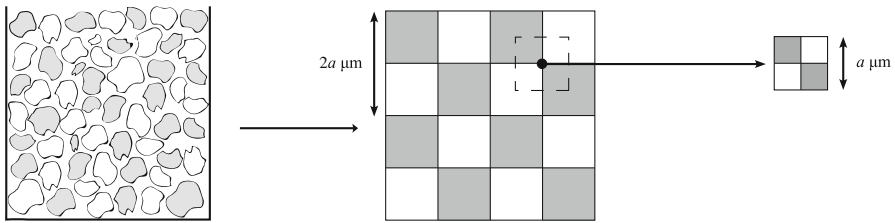
Fig. 1 Construction of a synthesis space in the two-dimensional two-reactant model: $V = (0, a) \times (0, a); a = 1 \, (\mu m)$
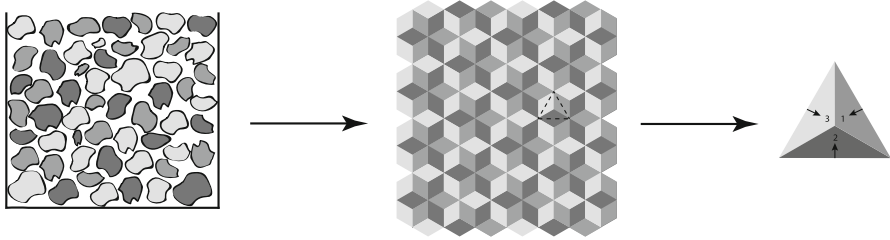

Fig. 2 Construction of a synthesis space in the two-dimensional three-reactant model

tively. So, in our calculations, we assume that the synthesis half-times are one-tenth of the corresponding "full" times, that is, 0.6, 0.8, 1.0, and 1.6 h at the temperatures $T = 1{,}200, 1{,}100, 1{,}000$, and $900 \, °C$, respectively.

At the preparatory stage of the synthesis the reactants are milled and mixed thoroughly in some cubic volume. The particles in the synthesis space are distributed randomly since their exact initial positions are unknown. As a result, modeling would require a huge memory size for storage. Therefore, we first assume that the particles are stored periodically. Moreover, in order to reduce the calculations to an area with zero Neumann boundary condition (zero normal derivative), we have to store the particles symmetrically with respect to any edge of the boundary of any particle. This is simple in the two-reactant case, as is shown in Fig. 1. In the three-reactant case, we achieve the periodicity and symmetry by taking rhombic particles so that the synthesis volume V is triangular as is shown in Fig. 2.

The typical volume of particles in the sol–gel synthesis method is $1 \, \mu m^3$. So, summarizing, we arrive at the equation system (5)–(8) in the triangular synthesis space V shown in Fig. 3, with the initial conditions in the three inner triangles proportional to the initial densities of the three reactants and zero Neumann boundary conditions, $\frac{\partial c_i}{\partial \mathbf{n}} = 0$ on ∂V, where $\mathbf{n}$ is the normal vector to the boundary.

When solving numerically the partial differential equation system (5)–(7) in the triangular area, we use a finite difference technique, which is rather complicated to be given here.

4 The main goal

Our estimations are based on the Arrhenius laws describing the temperature dependence of the diffusion and reaction rate coefficients by the following relations:

Fig. 3 The synthesis space in
the three-reactant model
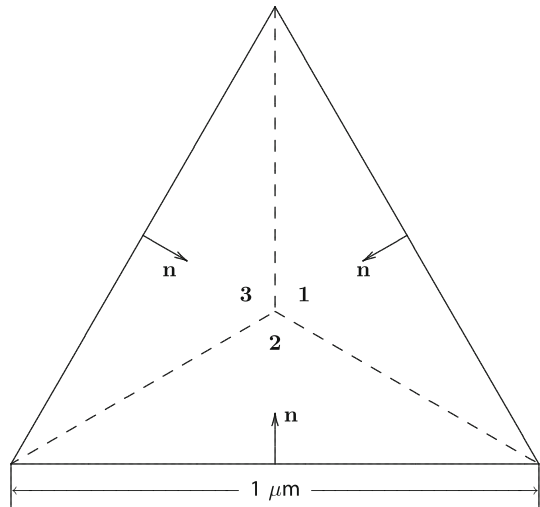
$$D = D_0 \exp\left\{-\frac{E_D}{RT}\right\}, \qquad k = k_0 \exp\left\{-\frac{E_A}{RT}\right\}. \tag{10}$$

Here, E_D is the diffusion activation energy, E_A is the reaction activation energy, D_0 and k_0 are constants, and $R = 8.314472$. Our main goal is to estimate the unknown parameters by finding the possible values of the coefficient pairs (D, k) for temperatures $T = 1{,}200, 1{,}100, 1{,}000,$ and $900\,°C$. In a sense, we solve the inverse problem for system (5)–(7): given the synthesis half-time $t_{1/2}$ at some temperature T, we look for the parameters D and k of the system such that the half-time condition (9) is satisfied. Then from the obtained data, using Eq. (10), we can find approximate values of the parameters E_D, E_A, D_0, k_0 and, in particular, the true values of parameters D and k for any given temperature T (see Sect. 5).

5 Calculation method

We calculate the diffusion and reaction rate coefficients as follows:

First, for the temperatures $T_1 = 1{,}200\,°C$, $T_2 = 1{,}100\,°C$, $T_3 = 1{,}000\,°C$, and $T_4 = 900\,°C$, we draw the corresponding graphs $L_1, L_2, L_3,$ and L_4 of points (D, k) of the diffusion and reaction rate parameters for which the half-times are equal to those of the laboratory experiments by using the sol–gel synthesis method (i.e., $t_{1/2} = 0.6, 0.8, 1,$ and 1.6, respectively). To this end, we use our computer program which, given any fixed D, half-time $t_{1/2}$, and particle size a, numerically solves the system (5)–(7) until the half-time condition (9) is satisfied and, using the middle-point method, finds the value k such that half-time coincides with the given one. For each temperature T, a sufficiently large discrete set of the (D, k) values is joined by a smooth curve. So, all four graphs, $L_1, L_2, L_3,$ and L_4, are shown in Fig. 4.

To calculate the four unknown parameters (E_D, E_A, D_0, k_0), we proceed as follows. For any pair $(D_1, k_1) \in L_1$ and any pair $(D_2, k_2) \in L_4$, using the Arrhenius law for
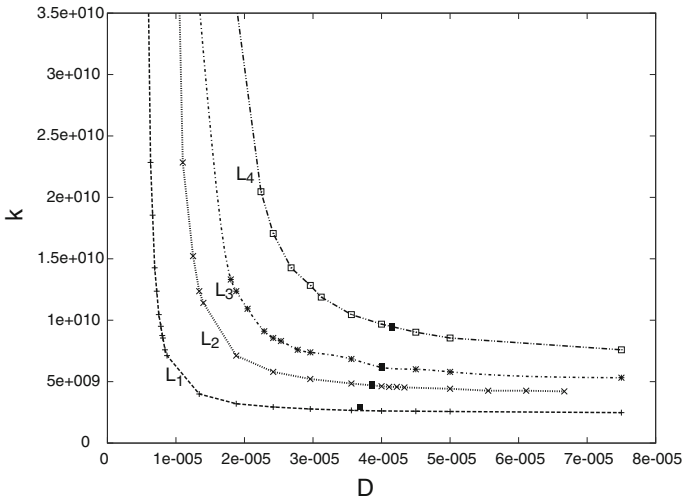
Fig. 4 Calculations in the three-reactant model in the triangular synthesis space V with edge $a = 1$ (μm) shown in Fig. 3

$T_1 = 1473$ K and $T_4 = 1173$ K, we write and solve the following equation system with respect to unknown (E_D, E_A, D_0, k_0):

$$D_1 = D_0 \exp \left\{ -\frac{E_D}{R \cdot T_1} \right\},$$

$$D_2 = D_0 \exp \left\{ -\frac{E_D}{R \cdot T_4} \right\},$$

$$k_1 = k_0 \exp \left\{ -\frac{E_A}{R \cdot T_1} \right\},$$

$$k_2 = k_0 \exp \left\{ -\frac{E_A}{R \cdot T_4} \right\}.$$

This way, we get the set of possible collections of the parameters $(E_D^i, E_A^i, D_0^i, k_0^i)$, $i = 1, 2, \ldots$. Repeating this for curves L_2 and L_3, we get another set of possible collections of the parameters, $(E_D^j, E_A^j, D_0^j, k_0^j)$, $j = 1, 2, \ldots$.

Using the weighted least-squares-method, we look for the minimal value of the sum

$$w_1 \left(E_D^i - E_D^j \right)^2 + w_2 \left(E_k^i - E_k^j \right)^2 + w_3 \left(D_0^i - D_0^j \right)^2 + w_4 \left(k_0^i - k_0^j \right)^2, \quad (11)$$

where we choose the weight coefficients w_i, $i = 1, 2, 3, 4$, so that the summands are approximately of the same order. The collections of parameters minimizing the sum (11), say $(E_D^{i0}, E_A^{i0}, D_0^{i0}, k_0^{i0})$ and $(E_D^{j0}, E_A^{j0}, D_0^{j0}, k_0^{j0})$, can both serve as estimates of (E_D, E_A, D_0, k_0). We finally take the average values of parameters: $(E_D, E_A, D_0, k_0) = ((E_D^{i0} + E_D^{j0})/2, (E_A^{i0} + E_A^{j0})/2, (D_0^{i0} + D_0^{j0})/2, (k_0^{i0} + k_0^{j0})/2)$. Now we can use these values to calculate D and k for arbitrary temperature T. Their values for temperatures T_i, $i = 1, 2, 3, 4$, are marked in Fig. 4 by "■".

Table 1 Diffusion and reaction rates

T	900 °C	1,000 °C	1,100 °C	1,200 °C
D	3.7e−5	3.9e−5	4e−5	4.1e−5
k	3e9	4.6e9	6.8e9	9.4e9

6 Calculation results

For the three-reactant model, we obtained the diffusion and reaction rates as functions of synthesis temperature T:

$$D = 6.5e5 - \exp(-5.6e3/(RT)), \tag{12}$$
$$k = 8.7e11 \exp(-5.5e4/(RT)). \tag{13}$$

The obtained Arrhenius law equations (12)–(13) give us the diffusion and reaction rates at temperatures $T = 900, 1,000, 1,100$, and $1,200$ °C, presented in Table 1.

7 Conclusions

Using a mathematical model based on second Fick's, active mass, and Arrhenius laws, we have analyzed a method for calculation of diffusion and reaction-rate coefficients of heterogeneous reactions at high temperatures in the two-dimensional three-reactant model. In comparison with the two-reactant case, the novelty of the model is that, in order to satisfy the periodicity and symmetry conditions on the form and distribution of the particles of three reactants, we consider rhombic-form particles distributed so that the calculations can be reduced to a triangular space. Having four half-times obtained by the same method at different temperatures:

1. We obtained explicit formulas for the coefficients expressing the dependence of the diffusion coefficient and reaction rate on the temperature, provided by Eqs. (12)–(13).
2. At the same time, we have calculated activation energies, important data that can be used to analyze other syntheses.
3. Using Eqs. (12)–(13), we have found the parameters D and k, and using computer modeling, the corresponding half-times for several temperatures.

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